Largest coefficients in binomial expansions

Milan Pahor
The University of New South Wales
<pahor@maths.unsw.edu.au>

The 1988 Extension 1 (3 unit) Mathematics New South Wales Higher School Certificate examination contained the following innovative question:

> Question 6 b)
>
> Suppose $(3x+7)^{25} = \sum_{k=0}^{25} t_k x^k$.
>
> i) Use the binomial theorem to write an expression for t_k, $0 \le k \le 25$.
>
> ii) Show that $\frac{t_{k+1}}{t_k} = \frac{3(25-k)}{7(k+1)}$.
>
> iii) Hence or otherwise find the largest coefficient t_k.
>
> You may leave your answer in the form $\binom{25}{k} 7^c 3^d$.

Similar questions have made regular subsequent appearances in trial examinations around NSW and many texts now devote whole chapters to the subtle and somewhat laborious process of establishing the largest coefficient in a binomial expansion. This article will produce a closed form solution to all questions of this type. Along the way, we will point out the limitations of the above approach and investigate the intriguing manner in which the greatest coefficient moves about.

The content of the article is accessible to advanced mathematics students in the final year of high school and is of particular value to Extension 1 and 2 students in the NSW Higher School Certificate. I have included a number of discussion points and extension problems to facilitate possible classroom discussions.

We begin our investigation by considering the simpler expansion

$$(2x + 1)^6 = 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$$

with

$$t_0 = 1,\ t_1 = 12,\ t_2 = 60,\ t_3 = 160,\ t_4 = 240,\ t_5 = 192 \text{ and } t_6 = 64.$$

The critical feature to note is that the coefficients rise and then fall in a controlled manner. That is

$$1 \le 12 \le 60 \le 160 \le 240 \ge 192 \ge 64.$$

The largest coefficient is clear with the coefficients first rising to and then falling from 240. This behaviour is in fact typical of certain binomial expansions and it is a property we exploit to attack larger questions where a direct expansion is impractical.

Comparing the ratio of each coefficient to its predecessor we have

$$\frac{t_1}{t_0}=\frac{12}{1}\ge 1,\ \frac{t_2}{t_1}=\frac{60}{12}\ge 1,\ \frac{t_3}{t_2}=\frac{160}{60}\ge 1,\ \frac{t_4}{t_3}=\frac{240}{160}\ge 1,\ \frac{t_5}{t_4}=\frac{192}{240}<1,\ \frac{t_6}{t_5}=\frac{64}{192}<1$$

If the largest k for which $\frac{t_{k+1}}{t_k}\ge 1$ is $k = m$ then the largest coefficient is t_{m+1}.

In the above expansion, the largest k for which $\frac{t_{k+1}}{t_k}\ge 1$ is $k = 3$ and thus the largest coefficient is $t_4 = 240$.

The general technique for establishing the maximal coefficient is built on this simple analysis of the ratio of successive binomial coefficients $\frac{t_{k+1}}{t_k}$.

[Discussion question: When attempting to maximise quantities we usually appeal to the calculus: why is it inappropriate to do so here?]

We will need a curious but very handy little function over the real line called the *floor function.*

Definition: floor(x) is the greatest integer less than or equal to x.

Thus: floor$(3.2) = 3$, floor$(-5.2) = -6$, floor$(\pi) = 3$ and floor$(2) = 2$.

The graph of the floor function is:

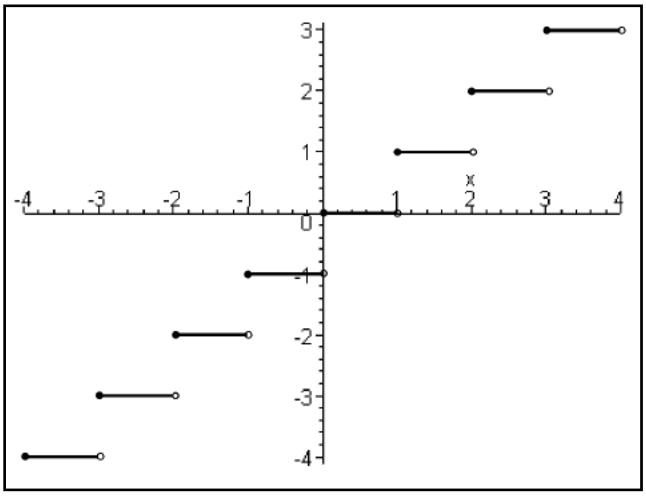

Figure 1

[Discussion Question: What is the domain and range of the floor function? Describe the domain, range and graph of the derivative of floor(x).]

We are now in a position to prove our first formula as a lemma.

Lemma

Suppose that d is a positive real number and that n is a positive integer. Then the largest coefficient in the binomial expansion of $(dx+1)^{n}$ is

$$\binom{n}{\alpha}d^{\alpha}$$

where

$$\alpha = \text{floor}\left(\frac{nd-1}{d+1}\right)+1$$

Proof

Using the Binomial theorem $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$ we have

$$(dx+1)^n = \sum_{k=0}^{n}\binom{n}{k}(dx)^k 1^{n-k} = \sum_{k=0}^{n}\binom{n}{k}d^k x^k = \sum_{k=0}^{n} t^k x^k$$

where $$t^k = \binom{n}{k}d^k,\ \ k = 0, 1\ldots\ n$$

Now

$$\frac{t_{k+1}}{t_k} = \frac{\binom{n}{k+1}d^{k+1}}{\binom{n}{k}d^k}$$

$$= \frac{\binom{n}{k+1}}{\binom{n}{k}}d$$

$$= \frac{\left(\frac{n!}{(k+1)!\left(n-(k+1)\right)!}\right)}{\left(\frac{n!}{k!(n-k)!}\right)}d$$

$$= \left(\frac{n!}{(k+1)!(n-k-1)}\right)\left(\frac{k!(n-k)!}{n!}\right)d$$

$$= \frac{n-k}{k+1}d$$

So $\frac{t_{k+1}}{t_k} \geq 1 \leftrightarrow \frac{n-k}{k+1}d \geq 1 \leftrightarrow nd - kd \geq k+1 \leftrightarrow nd-1 \geq kd+k \leftrightarrow k(d+1) \leq nd-1$

which is true if $k \le \dfrac{nd-1}{d+1}$ (noting that both $k+1$ and $d+1$ are positive).

Thus $\dfrac{t_{k+1}}{t_k} \ge 1$ only for k = 0, 1… floor$\left(\dfrac{nd-1}{d+1}\right)$.

Therefore the maximal coefficient is

$$t_\alpha = \binom{n}{\alpha} d^\alpha \quad \text{where } \alpha = \text{floor}\left(\frac{nd-1}{d+1}\right)+1$$

To complete the proof, note that

$$\frac{nd-1}{d+1} < \frac{nd+n}{d+1} < \frac{n(d+1)}{d+1} < n$$

and

$$\frac{nd-1}{d+1} > \frac{-d-1}{d+1} > -1$$

implying α is one of the integers 0, 1, 2… n.

Thus $t_\alpha = \binom{n}{\alpha} d^\alpha$ is well defined. QED.

Applying the above result to the expansion

$$(2x+1)^6 = 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$$

we have $d = 2$ and $n = 6$. Thus

$$\alpha = \text{floor}\left(\frac{nd-1}{d+1}\right)+1 = \text{floor}\left(\frac{12-1}{3}\right)+1 = \text{floor}(3.6)+1 = 4$$

The largest coefficient is then

$$t_\alpha = \binom{n}{\alpha} d^\alpha = \binom{6}{4} 2^4 = 15 \times 16 = 240$$

as expected!

Fixing n and varying d

It is fascinating to plot α versus d for a fixed n, say $n = 6$.
The sketch of $\alpha = \text{floor}\left(\dfrac{6d-1}{d+1}\right)+1$ is shown in Figure 2.

If we fix n and let d vary from 0 to n, the greatest coefficient in the expansion

$$(dx+1)^n = \sum_{k=0}^{n} t_k x^k$$

will elegantly march from the constant term x^0 up to the final term x^n.

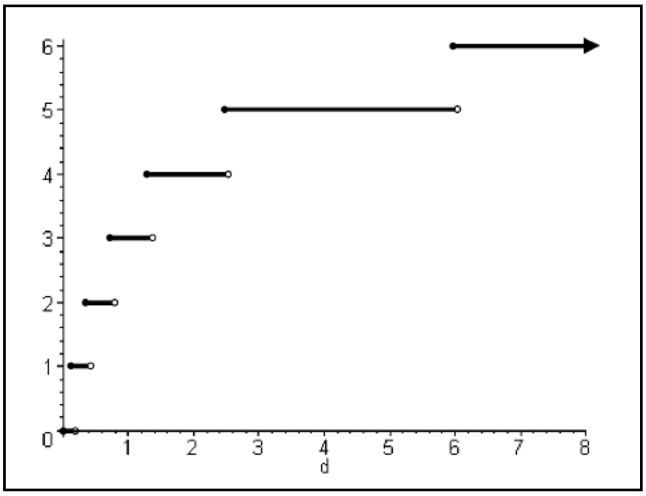

Figure 2

We have seen above that $0 \le \alpha \le n$. Furthermore

$$d = 0 \to \alpha = \text{floor}\left(\frac{nd-1}{d+1}\right) + 1 = \text{floor}\left(\frac{0-1}{0+1}\right) + 1 = \text{floor}(-1) + 1 = -1 + 1 = 0$$

and

$$d = n \to \alpha = \text{floor}\left(\frac{n^2-1}{n+1}\right) + 1 = \text{floor}\left(\frac{(n-1)(n+1)}{(n+1)}\right) + 1 = \text{floor}(n-1) + 1 = n - 1 + 1 = n$$

[Discussion question: Show that for fixed n, $f(d) = \dfrac{nd-1}{d+1}$ *is an increasing function of d.]*

Sampling a few expansions for $n = 6$ we have:

$(0.2x + 1)^6 = 1 + 1.2x + 0.6x^2 + 0.16x^3 + 0.024x^4 + 0.00192x^5 + 0.000064x^6$

$(x + 1)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

$(2x + 1)^6 = 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$

$(5x + 1)^6 = 1 + 30x + 375x^2 + 2500x^3 + 9375x^4 + 18750x^5 + 15625x^6$

The discontinuities in Figure 2 are of course the critical values of d where the largest coefficient in the expansion of $(dx + 1)^6$ leaps to its new home.

When does this happen?

We require $\dfrac{nd-1}{d+1} + 1 = m \quad (m = 1, 2 \ldots n)$. Thus:

$$\begin{aligned}
\frac{nd-1}{d+1} &= m - 1 \\
&\to nd - 1 = (m-1)(d+1) \\
&\to nd - 1 = md + m - d - 1 \\
&\to nd = md + m - d \\
&\to d(n - m + 1) = m \\
&\to d = \frac{m}{n-m+1}, \quad m = 1 \ldots n
\end{aligned}$$

It follows that the jumps occur at $d = \frac{1}{n}, \frac{2}{n-1}, \frac{3}{n-2} \cdots \frac{n-1}{2}, \frac{n}{1}$.

For $n = 6$, the critical values are

$$d = \frac{1}{6}, \frac{2}{5}, \frac{3}{4}, \frac{4}{3} \frac{5}{2}, \frac{6}{1}$$

Very cute and beautifully balanced! These are the six values of d where the greatest coefficient in the expansion of $(dx + 1)^6$ latches on to a new power of x. But how is the jump achieved? We have a few philosophical problems! The first is that the whole of binomial theory is characterised by symmetry and balance. What could possible justify one power of x over another at a critical value of d?

Furthermore for a particular power, say x^4 of x, its coefficient

$$t_4 = \binom{6}{4} d^2 = 15d^2$$

is a continuous function of d. How can these continuous evolutions lead to discrete jumps?

Well, let us take a look at the transition from

$(x + 1)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

to

$(2x + 1)^6 = 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$.

Somewhere between $d = 1$ and $d = 2$, the greatest coefficient has found its way from x^3 to x^4. From the above analysis this happens when $d = \frac{4}{3}$. The expansion at that point is:

$$\left(\frac{4}{3}x + 1\right)^6 = 1 + 8x + \frac{80}{3}x^2 + \frac{1280}{27}x^3 + \frac{1280}{27}x^4 + \frac{2048}{81}x^5 + \frac{4096}{729}x^6$$

The largest coefficient in the expansion

$$(dx+1)^n = \sum_{k=0}^{n} t_k x^k$$

slides along the expansion as d increases, *replicating itself* at the critical points

$$d = \frac{1}{n}, \frac{2}{n-1}, \frac{3}{n-2} \cdots \frac{n-1}{2}, \frac{n}{1}$$

where it needs to leap to the next power of x. A truly remarkable method of transport!

Before presenting our last result let us consider the role that the restriction that d be positive plays in this analysis. Consider the expansion:

$$(-3x + 1)^6 = 1 - 18x + 135x^2 - 540x^3 + 1215x^4 - 1458x^5 + 729x^6.$$

Observe that we no longer have the coefficients rising uniformly to a maximum and then falling! That is, the fundamental property driving our technique no longer holds. Note also that simply ignoring the negative will not work either since we will then get a greatest coefficient of 1458 rather than the correct answer of 1215.

[Discussion question: How could the above method be modified to properly deal with situations where $d < 0$?]

We are now in a position to state and prove our central result.

Theorem

Suppose that a and b are positive real numbers and n is a positive integer. Then the largest coefficient in the binomial expansion of $(ax + b)^n$ is

$$\binom{n}{\alpha} a^{\alpha} b^{n-\alpha} \quad \text{where } \alpha = \text{floor}\left(\frac{na-b}{a+b}\right)+1$$

Proof

$$\left(ax+b\right)^n = \left(b\left(\frac{a}{b}x+1\right)\right)^n = b^n\left(dx+1\right)^n \quad \text{where } d = \frac{a}{b}$$

All coefficients are multiplied by b^n so by the above lemma, the largest coefficient is given by

$$b^n\binom{n}{\alpha}d^{\alpha} = b^n\binom{n}{\alpha}\left(\frac{a}{b}\right)^{\alpha} = \binom{n}{\alpha}a^{\alpha}b^{n-\alpha}$$

where

$$\alpha = \text{floor}\left(\frac{nd-1}{d+1}\right)+1 = \text{floor}\left(\frac{n\left(\frac{a}{b}-1\right)}{\left(\frac{a}{b}+1\right)}\right)+1 = \text{floor}\left(\frac{na-b}{a+b}\right)+1$$

as required.
QED.

Returning to our original HSC question regarding the expansion of $(3x + 7)^{25}$ we have $a = 3$, $b = 7$, and $n = 25$. Thus

$$\alpha = \text{floor}\left(\frac{na-b}{a+b}\right)+1 = \text{floor}\left(\frac{25\times 3-7}{3+7}\right)+1 = \text{floor}\left(6.8\right)+1 = 7$$

The largest coefficient is therefore

$$\binom{n}{\alpha}a^{\alpha}b^{n-\alpha} = \binom{25}{7}3^7 7^{25-7} = \binom{25}{7}3^7 7^{18}$$

[Investigation: Try to establish a formula for the greatest coefficient in the expansion of $(ax^p + bx^q)^n$. What restrictions do you need to place on a, b, p and q?]